

# Periodicity of cylindrical linear cellular automata

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## Abstract

Periodicity and relaxation are investigated for the trajectories of the states in cylindrical linear cellular automata. The time evolutions are described with matrices. The eigenvalue analysis is applied to obtain the maximum values of period and relaxation. The translational invariance suppresses the maximum period and relaxation.

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## I. INTRODUCTION

Cellular automata (CA) are one of the simplest mathematical models for nonlinear dynamics to produce complex patterns of behavior. They had been originally introduced by von Neumann [1] to investigate some artificial life. Wolfram had reintroduced CA as a model to investigate complexity and randomness [2]. He investigated many fundamental features of them [3–5]. Since then many authors have made efforts to clarify the properties of CA and applied to natural systems including biological systems, complex fluids, chemical reactions, astronomical systems and so on [6].

Computational simplicity is the advantages of using CA to simulate complex behaviors. Discrete states are defined on each lattice point and evolve with discrete time steps in CA. Moreover nonlinear properties are expected to be emphasized by discretization. A simple modeling, however, does not mean simple properties. Even though the great efforts, the fundamental properties of CA are still unclear. Studies on these simple systems seems to be a key for understanding discretized simulated systems.

One-dimensional CA are described by the discrete time evolution of site  $a_i$ :

$$a_i(t+1) = F[a_{i-r}(t), a_{i-r+1}(t), \dots, a_i(t), \dots, a_{i+r}(t)], \quad (1.1)$$

where  $a_i$  takes  $k$  discrete values over  $Z_k$ . The simplest models, *elementary cellular automata*, consist of sites with two internal states over  $Z_2$  ( $k = 2$ ) interacting with the nearest neighbor sites ( $r = 1$ ). Wolfram introduced a naming scheme for these models and classified the behavior of CA into four classes [2,3].

Most authors have worked on CA within the scope of the infinite number of sites. A few works have concerned the effects of finiteness. The orbits of *cylindrical* linear CA, which are CA with periodic boundaries, are analyzed on the basis of *characteristic polynomials*, which directly describe the states of linear CA [5,7–10]. Stevens, Rosensweig and Cerkowicz had investigated rule 90 CA, which will be described in II, with Dirichlet boundary conditions. They analyzed the eigenvalue polynomials of the matrices which give the time evolution of

the system [11]. In our previous papers [12,13] (referred as papers I and II), we had also investigated the periodic orbits of finite linear CA (rule 90 and 150) with Dirichlet boundary conditions by analyzing the eigenvalue equations. In the present paper the method is applied to cylindrical linear CA. The proof of the classification of the orbits for Dirichlet boundary cases is given in Appendix A.

## II. THE MODELS AND ORBITS OF STATES

There are two examples of linear cellular automata in the elementary ones ( $k = 2, r = 1$ ). They are called as rule 90 and rule 150 following the Wolfram's naming scheme. For rule 90 CA, the time evolution of the  $i$ -th site  $a_i \in \{0, 1\}$  ( $i = 1, \dots, N$ ) is described as a sum modulo 2 of the nearest-neighbor sites:

$$a_i(t+1) = a_{i-1}(t) + a_{i+1}(t) \bmod 2. \quad (2.1)$$

The time evolution of rule 150 CA is given as a sum modulo 2 of the nearest-neighbor sites and itself:

$$a_i(t+1) = a_{i-1}(t) + a_i(t) + a_{i+1}(t) \bmod 2. \quad (2.2)$$

We use the periodic boundary conditions:  $a_0 = a_N, a_{N+1} = a_1$ . These models belong to the third class which show the chaotic behavior in the Wolfram's classification

These models are linear because the rules are additive and the time evolutions are also expressed by the matrices

$$A(t+1) = UA(t), \quad (2.3)$$

where  $A(t) = {}^t(a_1(t), a_2(t), \dots, a_N(t))$  describes the state,  $N$  bits binary number, at  $t$ . The components of the transfer matrices  $U$  are given by

$$U_{ij} = \begin{cases} 1 & j = i \pm 1 \\ 1 & i = 1, j = N \\ 1 & i = N, j = 1 \\ 0 & \text{otherwise} \end{cases} \quad (\text{rule-90}), \quad (2.4)$$

$$= \begin{cases} 1 & j = i \pm 1 \\ 1 & j = i \\ 1 & i = 1, j = N \quad (\text{rule-150}). \\ 1 & i = N, j = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.5)$$

The trajectories of the states can be classified into three cases with the properties of the transfer matrices as mentioned in papers I and II (See Fig. 1).

$$U^{\Pi_N} = I, \quad (2.6)$$

$$U^{\Pi_N + \pi_N} = U^{\pi_N}, \quad (2.7)$$

$$U^{f_N} = 0, \quad (2.8)$$

where  $I$  denotes a unit matrix. The first case, Eq. (2.6), corresponds to perfectly periodic motions with the maximum period  $\Pi_N$ . Namely all states belong to perfectly periodic orbits whose period does not exceed  $\Pi_N$ . The second case, Eq. (2.7), shows periodic motions with relaxation. A relaxation path whose time step is not greater than  $\pi_N$  leads to a periodic orbit with the maximum period  $\Pi_N$ . The third, Eq. (2.8), is a special case of the second one. All states are drawn to a null state on the contrary to that the current CA belong to the third class of Wolfram's classification of motions (chaotic motions).

### III. EIGENVALUE ANALYSIS

The eigenvalue polynomials for  $N$ -site linear cellular automata are defined by

$$D_N^{(R)}(\lambda) = |U + \lambda I|, \quad (3.1)$$

where the index  $R$  specifies the rule number and the boundary condition, for example, 90D for rule 90 CA with Dirichlet boundaries and 150P for rule 150 with periodic boundaries. Since each site  $a_i$  takes binary values, the eigenvalue polynomials are over  $Z_2$ , namely each coefficient is 0 or 1. The eigenvalues  $\lambda$  are not usual numbers but over the Galois field  $\text{GF}(2^N)$ , a finite field with  $2^N$  elements.

The eigenvalue polynomials enable us to find the maximum period  $\Pi_N$  and the maximum relaxation  $\pi_N$  as discussed in Ref. [11], papers I and II. For nilpotent cases,  $D_N^{(R)}(\lambda) = \lambda^N$ , all states are drawn into a null state within  $N$  steps or less. If the eigenvalue polynomial has a constant term,  $D_N^{(R)}(0) = 1$ , the eigenvalue equation  $D_N^{(R)}(\lambda) = 0$  can be reduced to a simple form as  $\lambda^{P_N} + 1 = 0$  by multiplying some powers of  $\lambda$  and repeatedly substituting the eigenvalue equation [14]. Then the minimum value of  $P_N$  corresponds to the maximum period  $\Pi_N$ . In other words the maximum period  $\Pi_N$  is the minimum integer  $m$  satisfying  $D_N^{(R)}(\lambda) | (\lambda^m - 1)$ , where the notation  $f|g$  denotes  $f$  divides  $g$ . The remaining cases are mixture of above two cases as  $D_N^{(R)}(\lambda) = \lambda^{p_N} \times d(\lambda)$ , where the polynomial  $d(\lambda)$  has a constant term ( $D_N^{(R)}(0) = 0$  and  $d(0) = 1$ ). One can evaluate the maximum period with the above procedure from  $d(\lambda)$ . The maximum period is the minimum integer  $m$  satisfying  $d(\lambda) | (\lambda^m - 1)$ . The relaxation length to periodic orbits is  $p_N$ . Actual periods and relaxation paths are given by divisors of the maximum values respectively. They depend on the symmetry of the initial states as shown in papers I and II.

The eigenvalue polynomials for rule 90 cylindrical CA obey the relation

$$D_N^{(90P)}(\lambda) = \lambda D_{N-1}^{(90D)}(\lambda). \quad (3.2)$$

Orbits of rule 90 cylindrical CA can be classified as

$$U^{\Pi_N+1} = U \quad (N \text{ is odd}), \quad (3.3)$$

$$U^{\Pi_N+\pi_N} = U^{\pi_N} \quad (N \text{ is even except } N = 2^n), \quad (3.4)$$

$$U^N = 0 \quad (N = 2^n), \quad (3.5)$$

by Eq. (3.2) and the classification of orbits for rule 90 finite CA with Dirichlet boundaries (see Appendix A). There are no perfect periodic orbits by comparing with the Dirichlet boundary cases.

The maximum values of the period and relaxation are derived from the polynomials

$$D_N^{(90P)}(\lambda) = \sum_{j=0}^{\lfloor (N-1)/2 \rfloor} \left( C_j^{N-1} \bmod 2 \right) \lambda^{N-2j}, \quad (3.6)$$

which are the solutions of Eq. (3.2), where  $\lfloor x \rfloor$  denotes the largest integer not exceeding  $x$  and

$$C_j^N = (-1)^j \binom{N-j}{j}. \quad (3.7)$$

Some examples of the eigenvalue polynomials are shown in Table I. Equation (3.2) leads that the maximum period for rule 90 cylindrical CA with  $N$  sites are equal to that for  $N-1$ -site case with Dirichlet boundaries,  $\Pi_N(\text{rule 90P}) = \Pi_{N-1}(\text{rule 90D})$ .

The eigenvalue polynomials for rule 150 cylindrical CA can be also written with those with Dirichlet boundaries

$$D_N^{(150P)}(\lambda) = (1 + \lambda) D_{N-1}^{(150D)}(\lambda). \quad (3.8)$$

Orbits can be classified as

$$U^{\Pi_N} = I \quad (N \neq 3n), \quad (3.9)$$

$$U^N = I \quad (N = 2^n), \quad (3.10)$$

$$U^{\Pi_N + \pi_N} = U^{\pi_N} \quad (N = 3n). \quad (3.11)$$

by virtue of the classification of orbits for rule 150 CA with Dirichlet boundaries. There are no characteristic difference from the Dirichlet boundary cases. No simple relations on the maximum periods between the periodic and Dirichlet boundary cases are found.

The maximum periods and relaxations are derived from the polynomials

$$D_N^{(150P)}(\lambda) = \left( \sum_{j=0}^{\lfloor (N-1)/2 \rfloor} C_j^{N-1} \bmod 2 \right) + \sum_{k=1}^N \left( \sum_{j=0}^{\lfloor (N-k)/2 \rfloor} C_j^{N-1} \binom{N-2j}{k} \bmod 2 \right) \lambda^k. \quad (3.12)$$

Some examples of the eigenvalue polynomials are shown in Table II.

The actual period and relaxation of the system may be suppressed by the translational symmetry which the periodic boundary conditions ensure. Direct matrix multiplications are also the tool to find the maximum period and relaxation. The results are summarized in Table III with those obtained from the eigenvalue analysis. The suppressed maximum period, which means the maximum period obtained by the direct matrix multiplications,

are just a half of those by the eigenvalue analysis. On the relaxation, the same situations happen except for the  $\pi_N = 1$  cases. The suppressed maximum periods are plotted in Figs. 2 and 3.

As the results from the suppression of the maximum values of the period and relaxation, we can find a simple relation of the period of rule 90 CA with periodic and Dirichlet boundaries. The period of  $N$ -site rule 90 cylindrical CA is half of  $N - 1$ -site rule 90 CA with Dirichlet boundaries. No such simple relations are found for rule 150 cases.

#### IV. SUMMARY AND DISCUSSION

The method of the eigenvalue analysis was applied to linear cellular automata with periodic boundaries. The trajectories of the states of CA were classified. The maximum values of the period and relaxation were evaluated by the eigenvalue equations. For rule 90 cases a simple relation was found between the eigenvalue polynomials with periodic boundaries and those with Dirichlet boundaries.

The cylindrical CA enjoy the translational invariance with periodic boundaries. This translational symmetry suppresses the maximum values of the period and relaxation.

In the previous papers [12,13] I had found the symmetry of the initial states suppress the period. So it is natural that the translational symmetry suppresses the period and relaxation. The translational symmetry changes the form of the transfer matrices  $U$  and consequently the form of the eigenvalue polynomials. The eigenvalue polynomials, however, do not tell us the suppression of the period and relaxation.

The eigenvalue analysis may not be able to handle the translational invariance well. The matrix  $U$  for rule 90 CA can be decomposed as  $U = L + R$ , where  $L$  and  $R$  denote the translation operation to the left and right direction respectively. For rule 150 CA it can be describe as  $U = L + R + I$ . Namely the dynamics is described only with the composition of the translation operation. This may be the reason of the different results from the eigenvalue analysis and the direct matrix multiplications.

Let me investigate the cas of  $N = 4$  rule 90 CA with periodic boundary conditions for instance. The eigenvalue equation is  $\lambda^4 = 0$ . This can be expressed with the transfer matrix  $U$  as

$$U^4 = (L + R)^4 = L^4 + R^4 = I + I = 0, \quad (4.1)$$

where all operations are carried over GF(2). On the other hand, another identity

$$U^2 = (L + R)^2 = L^2 + R^2 = 0 \quad (4.2)$$

holds by virtue of the identity

$$L^m = R^{N-m}. \quad (4.3)$$

This identity may not be taken account into the eigenvalue analysis.

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## APPENDIX A: CLASSIFICATION OF ORBITS FOR DIRICHLET BOUNDARY CASES

I had given the classification of orbits of rule 90 and rule 150 finite linear cellular automata with Dirichlet boundaries in paper II. Here I give brief proofs of the classification.

For rule 90 cases, the eigenvalue polynomials obey the following recursion relation:

$$D_N^{(90D)}(\lambda) = \lambda D_{N-1}^{(90D)}(\lambda) - D_{N-2}^{(90D)}(\lambda). \quad (A1)$$

The equation (A1) is simplified for  $\lambda = 0$  case to  $D_N^{(90D)}(0) = D_{N-2}^{(90D)}(0)$ . With the special values  $D_3^{(90D)}(0) = 0$  and  $D_4^{(90D)}(0) = 1$ , one can find simple relations:

$$D_{2n}^{(90D)}(0) = D_{2n-2}^{(90D)}(0) = D_4^{(90D)}(0) = \dots = 1, \quad (\text{A2})$$

$$D_{2n+1}^{(90D)}(0) = D_{2n-1}^{(90D)}(0) = D_3^{(90D)}(0) = \dots = 0. \quad (\text{A3})$$

The eigenvalue polynomials for even  $N$  have a constant term (Eq. (A2)). This means that rule 90 CA with Dirichlet boundaries with even number of sites show perfectly periodic motions. On the contrary, Eq. (A3) shows that rule 90 CA with Dirichlet boundaries with odd number of sites have periodic orbits with relaxation paths or paths drawn into a null state.

The another recursion relation

$$D_N^{(90D)}(\lambda) = \lambda^2 D_{N-2}^{(90D)}(\lambda) + D_{N-4}^{(90D)}(\lambda) \quad (\text{A4})$$

is derived from Eq. (A1). By solving Eq. (A4) with  $D_3^{(90D)}(\lambda) = \lambda^3$  and  $D_5^{(90D)}(\lambda) = \lambda^5 + \lambda$ , a simple relation

$$D_{2n+1}^{(90D)}(\lambda) = \lambda D_n^{(90D)}(\lambda^2) \quad (\text{A5})$$

is derived [15]. This gives explicit expressions of the eigenvalue polynomials for  $N = 2^n - 1$  cases:

$$D_{2^n-1}^{(90D)}(\lambda) = \lambda D_{2^{n-1}-1}^{(90D)}(\lambda^2) = \dots = \lambda^{2^n-1}. \quad (\text{A6})$$

On the other hand, Eq. (A1) gives the explicit expressions of the eigenvalue polynomials with binomial coefficients as

$$D_{2^n-1}^{(90D)}(\lambda) = \sum_{j=0}^{2^n-1} \left( \binom{2^n-1-j}{j} \bmod 2 \right) \lambda^{2^n-1-2j}. \quad (\text{A7})$$

Comparing Eqs. (A6) and (A7), identities on binomial coefficients

$$\binom{2^n-1-j}{j} \bmod 2 = 0 \quad (j \neq 0) \quad (\text{A8})$$

are obtained. The eigenvalue polynomials for odd  $N (\neq 2^n - 1)$  can be expressed with those for even  $N$  by repeated usages of Eq. (A5). So they can not be nilpotent. Therefore the following classification of orbits

$$U^{\Pi_N} = I \quad (N \text{ is even}), \quad (\text{A9})$$

$$U^{\Pi_N + \pi_N} = U^{\pi_N} (N \text{ is odd} (\neq 2^n - 1)), \quad (\text{A10})$$

$$U^N = 0 \quad (N = 2^n - 1), \quad (\text{A11})$$

is proven.

For rule 150 cases, the recursion relation changes to

$$D_N^{(150D)}(\lambda) = (1 + \lambda)D_{N-1}^{(150D)}(\lambda) - D_{N-2}^{(150D)}(\lambda). \quad (\text{A12})$$

This leads to a simple relation  $D_N^{(150D)}(0) = D_{N-3}^{(150D)}(0)$  with  $D_3^{(150D)}(0) = D_4^{(150D)}(0) = 1$  and  $D_5^{(150D)}(0) = 0$ . Therefore one can obtain simple relations

$$D_{3n}^{(150D)}(0) = \dots = D_3^{(150D)}(0) = 1, \quad (\text{A13})$$

$$D_{3n+1}^{(150D)}(0) = \dots = D_4^{(150D)}(0) = 1, \quad (\text{A14})$$

$$D_{3n+2}^{(150D)}(0) = \dots = D_5^{(150D)}(0) = 0. \quad (\text{A15})$$

Orbits of rule 150 finite linear CA are perfectly periodic except  $N = 3n + 2$  cases.

One can prove that the eigenvalue polynomials of rule 150 CA with  $N = 3n + 2$  are not nilpotent for  $n > 0$ . Corresponding to Eqs. (A4) and (A5), recursion relations

$$D_N^{(150D)}(\lambda) = (1 + \lambda^2)D_{N-2}^{(150D)}(\lambda) + D_{N-4}^{(150D)}(\lambda), \quad (\text{A16})$$

$$D_{2n+1}^{(150D)}(\lambda) = (1 + \lambda)D_n^{(150D)}(\lambda^2), \quad (\text{A17})$$

hold. Applying them to odd  $3n + 2$ , namely odd  $n$ , one can obtain the new recursion relation

$$D_{3n+2}^{(150D)}(\lambda) = (1 + \lambda)D_{3((n-1)/2)+2}^{(150D)}(\lambda^2). \quad (\text{A18})$$

By the special case  $D_5^{(150D)}(\lambda) = \lambda^5 + \lambda^4$ ,  $D_{3n+2}^{(150D)}(\lambda)$  can not be nilpotent for odd  $n$ . For the even  $3n + 2$  cases, the new recursion relation

$$D_N^{(150D)}(\lambda) = \lambda^4 D_{N-4}^{(150D)}(\lambda) + (1 + \lambda^2)D_{N-6}^{(150D)}(\lambda) \quad (\text{A19})$$

is used. The lowest order term of  $D_8^{(150D)}(\lambda) = \lambda^8 + \lambda^6 + \lambda^2$  is  $\lambda^2$ . And if  $D_{3n+2}^{(150D)}(\lambda)$  has a  $\lambda^2$  as the lowest order,

$$D_{3n+8}^{(150D)}(\lambda) = \lambda^4 D_{3n+4}^{(150D)}(\lambda) + (1 + \lambda^2) D_{3n+2}^{(150D)}(\lambda) \quad (\text{A20})$$

tells that also the lowest order of  $D_{3n+8}^{(150D)}(\lambda)$  is  $\lambda^2$ . Namely  $D_{3n+2}^{(150D)}(\lambda)$  must have  $\lambda^{3n+2}$  and  $\lambda^2$  terms at least and can not be nilpotent for even  $3n+2$  ( $n > 2$ ).

Let me investigate special cases with  $N = 2^n - 1$ . The recursion relation Eq. (A12) gives

$$D_N^{(150D)}(\lambda) = \sum_{k=0}^N \left( \sum_{j=0}^{\lfloor (N-k)/2 \rfloor} \binom{N-j}{j} \binom{N-2j}{k} \bmod 2 \right) \lambda^k. \quad (\text{A21})$$

For special cases  $N = 2^n - 1$ , this reduces to

$$D_{2^n-1}^{(150D)}(\lambda) = \sum_{k=0}^N \lambda^k, \quad (\text{A22})$$

with the identity Eq. (A8) and  $\binom{2^n-1}{k} \bmod 2 = 1$  ( $2^n - 1 \geq k \geq 0$ ). This gives explicit expressions of the periods as  $\Pi_{2^n-1} = 2^n$ . Therefore the classification of orbits for rule 150 CA with Dirichlet boundaries

$$U^{\Pi_N} = I \quad (N \neq 3n+2), \quad (\text{A23})$$

$$U^N = I \quad (N = 2^n - 1), \quad (\text{A24})$$

$$U^{\Pi_N + \pi_N} = U^{\pi_N} \quad (N = 3n+2), \quad (\text{A25})$$

is proven for  $N > 2$ .

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[15] Equation (A5) gives simple relation of the maximum period and relaxation as  $\Pi_{2n+1} = 2\Pi_n$  and  $\pi_{2n+1} = 2\pi_n + 1$  (there are mistakes in Refs. [12] and [13]).

## FIGURES

FIG. 1. Schematic features of the trajectories of cellular automata: (a) simple periodic orbit, (b) periodic orbit with relaxation, (c) limit point.

FIG. 2. Actual period for rule 90 cylindrical cellular automata

FIG. 3. Actual period for rule 150 cylindrical cellular automata

# TABLES

TABLE I. Eigenvalue polynomials of rule 90 cylindrical cellular automata

4	$\lambda^4$
5	$\lambda^5 + \lambda^3 + \lambda^1$
6	$\lambda^6 + \lambda^2$
7	$\lambda^7 + \lambda^5 + \lambda^1$
8	$\lambda^8$
9	$\lambda^9 + \lambda^7 + \lambda^5 + \lambda^1$
10	$\lambda^{10} + \lambda^6 + \lambda^2$
11	$\lambda^{11} + \lambda^9 + \lambda^5 + \lambda^3 + \lambda^1$
12	$\lambda^{12} + \lambda^4$
13	$\lambda^{13} + \lambda^{11} + \lambda^9 + \lambda^3 + \lambda^1$
14	$\lambda^{14} + \lambda^{10} + \lambda^2$
15	$\lambda^{15} + \lambda^{13} + \lambda^9 + \lambda^1$
16	$\lambda^{16}$
17	$\lambda^{17} + \lambda^{15} + \lambda^{13} + \lambda^9 + \lambda^1$
18	$\lambda^{18} + \lambda^{14} + \lambda^{10} + \lambda^2$
19	$\lambda^{19} + \lambda^{17} + \lambda^{13} + \lambda^{11} + \lambda^9 + \lambda^3 + \lambda^1$
20	$\lambda^{20} + \lambda^{12} + \lambda^4$
21	$\lambda^{21} + \lambda^{19} + \lambda^{17} + \lambda^{11} + \lambda^9 + \lambda^5 + \lambda^3 + \lambda^1$
22	$\lambda^{22} + \lambda^{18} + \lambda^{10} + \lambda^6 + \lambda^2$
23	$\lambda^{23} + \lambda^{21} + \lambda^{17} + \lambda^9 + \lambda^7 + \lambda^5 + \lambda^1$
24	$\lambda^{24} + \lambda^8$
25	$\lambda^{25} + \lambda^{23} + \lambda^{21} + \lambda^{17} + \lambda^7 + \lambda^5 + \lambda^1$
26	$\lambda^{26} + \lambda^{22} + \lambda^{18} + \lambda^6 + \lambda^2$
27	$\lambda^{27} + \lambda^{25} + \lambda^{21} + \lambda^{19} + \lambda^{17} + \lambda^5 + \lambda^3 + \lambda^1$
28	$\lambda^{28} + \lambda^{20} + \lambda^4$

29	$\lambda^{29} + \lambda^{27} + \lambda^{25} + \lambda^{19} + \lambda^{17} + \lambda^3 + \lambda^1$
30	$\lambda^{30} + \lambda^{26} + \lambda^{18} + \lambda^2$
31	$\lambda^{31} + \lambda^{29} + \lambda^{25} + \lambda^{17} + \lambda^1$
32	$\lambda^{32}$

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TABLE II. Eigenvalue polynomials of rule 150 cylindrical cellular automata

3	$\lambda^3 + \lambda^2$
4	$\lambda^4 + 1$
5	$\lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda^1 + 1$
6	$\lambda^6 + \lambda^4$
7	$\lambda^7 + \lambda^6 + \lambda^3 + \lambda^2 + \lambda^1 + 1$
8	$\lambda^8 + 1$
9	$\lambda^9 + \lambda^8 + \lambda^7 + \lambda^6 + \lambda^3 + \lambda^2$
10	$\lambda^{10} + \lambda^8 + \lambda^6 + \lambda^4 + \lambda^2 + 1$
11	$\lambda^{11} + \lambda^{10} + \lambda^5 + \lambda^4 + \lambda^1 + 1$
12	$\lambda^{12} + \lambda^8$
13	$\lambda^{13} + \lambda^{12} + \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^5 + \lambda^4 + \lambda^1 + 1$
14	$\lambda^{14} + \lambda^{12} + \lambda^6 + \lambda^4 + \lambda^2 + 1$
15	$\lambda^{15} + \lambda^{14} + \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^7 + \lambda^6 + \lambda^3 + \lambda^2$
16	$\lambda^{16} + 1$
17	$\lambda^{17} + \lambda^{16} + \lambda^{15} + \lambda^{14} + \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^7 + \lambda^6 + \lambda^3 + \lambda^2 + \lambda^1 + 1$
18	$\lambda^{18} + \lambda^{16} + \lambda^{14} + \lambda^{12} + \lambda^6 + \lambda^4$
19	$\lambda^{19} + \lambda^{18} + \lambda^{13} + \lambda^{12} + \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda^1 + 1$
20	$\lambda^{20} + \lambda^{16} + \lambda^{12} + \lambda^8 + \lambda^4 + 1$
21	$\lambda^{21} + \lambda^{20} + \lambda^{19} + \lambda^{18} + \lambda^{17} + \lambda^{16} + \lambda^{11} + \lambda^{10} + \lambda^3 + \lambda^2$
22	$\lambda^{22} + \lambda^{20} + \lambda^{10} + \lambda^8 + \lambda^2 + 1$
23	$\lambda^{23} + \lambda^{22} + \lambda^{19} + \lambda^{18} + \lambda^{17} + \lambda^{16} + \lambda^9 + \lambda^8 + \lambda^1 + 1$
24	$\lambda^{24} + \lambda^{16}$
25	$\lambda^{25} + \lambda^{24} + \lambda^{23} + \lambda^{22} + \lambda^{19} + \lambda^{18} + \lambda^9 + \lambda^8 + \lambda^1 + 1$
26	$\lambda^{26} + \lambda^{24} + \lambda^{22} + \lambda^{20} + \lambda^{18} + \lambda^{16} + \lambda^{10} + \lambda^8 + \lambda^2 + 1$
27	$\lambda^{27} + \lambda^{26} + \lambda^{21} + \lambda^{20} + \lambda^{17} + \lambda^{16} + \lambda^{11} + \lambda^{10} + \lambda^3 + \lambda^2$
28	$\lambda^{28} + \lambda^{24} + \lambda^{12} + \lambda^8 + \lambda^4 + 1$

29	$\lambda^{29} + \lambda^{28} + \lambda^{27} + \lambda^{26} + \lambda^{25} + \lambda^{24} + \lambda^{21} + \lambda^{20} + \lambda^{17} + \lambda^{16} + \lambda^{13} + \lambda^{12}$ $+ \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + \lambda^1 + 1$
30	$\lambda^{30} + \lambda^{28} + \lambda^{22} + \lambda^{20} + \lambda^{18} + \lambda^{16} + \lambda^{14} + \lambda^{12} + \lambda^6 + \lambda^4$
31	$\lambda^{31} + \lambda^{30} + \lambda^{27} + \lambda^{26} + \lambda^{25} + \lambda^{24} + \lambda^{23} + \lambda^{22}$ $+ \lambda^{19} + \lambda^{18} + \lambda^{15} + \lambda^{14} + \lambda^{11} + \lambda^{10} + \lambda^9 + \lambda^8 + \lambda^7 + \lambda^6$ $+ \lambda^3 + \lambda^2 + \lambda^1 + 1$
32	$\lambda^{32} + 1$

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TABLE III. Period for rule-90 and 150 cylindrical CA.

$N$	Rule-90		Rule-150	
	Polynomials	Actual	Polynomials	Actual
4	$U^4 = 0$	$U^2 = 0$	$U^4 = I$	$U^2 = I$
5	$U^7 = U$	$U^4 = U$	$U^6 = I$	$U^3 = I$
6	$U^6 = U^2$	$U^3 = U$	$U^6 = U^4$	$U^3 = U^2$
7	$U^{15} = U$	$U^8 = U$	$U^{14} = I$	$U^7 = I$
8	$U^8 = 0$	$U^4 = 0$	$U^8 = I$	$U^4 = I$
9	$U^{15} = U$	$U^8 = U$	$U^{16} = U^2$	$U^8 = U$
10	$U^{14} = U^2$	$U^7 = U$	$U^{12} = I$	$U^6 = I$
11	$U^{63} = U$	$U^{32} = U$	$U^{62} = I$	$U^{31} = I$
12	$U^{12} = U^4$	$U^6 = U^2$	$U^{12} = U^8$	$U^6 = U^4$
13	$U^{127} = U$	$U^{64} = U$	$U^{42} = I$	$U^{21} = I$
14	$U^{30} = U^2$	$U^{15} = U$	$U^{28} = I$	$U^{14} = I$
15	$U^{31} = U$	$U^{16} = U$	$U^{32} = U^2$	$U^{16} = U$
16	$U^{16} = 0$	$U^8 = 0$	$U^{16} = I$	$U^8 = I$
17	$U^{31} = U$	$U^{16} = U$	$U^{30} = I$	$U^{15} = I$
18	$U^{30} = U^2$	$U^{15} = U$	$U^{32} = U^4$	$U^{16} = U^2$
19	$U^{1023} = U$	$U^{512} = U$	$U^{1022} = I$	$U^{511} = I$
20	$U^{28} = U^4$	$U^{14} = U^2$	$U^{24} = I$	$U^{12} = I$
21	$U^{127} = U$	$U^{64} = U$	$U^{128} = U^2$	$U^{64} = U$
22	$U^{126} = U^2$	$U^{63} = U$	$U^{124} = I$	$U^{62} = I$
23	$U^{4095} = U$	$U^{2048} = U$	$U^{4094} = I$	$U^{2047} = I$
24	$U^{24} = U^8$	$U^{12} = U^4$	$U^{24} = U^{16}$	$U^{12} = U^8$
25	$U^{2047} = U$	$U^{1024} = U$	$U^{2046} = I$	$U^{1023} = I$
26	$U^{254} = U^2$	$U^{127} = U$	$U^{84} = I$	$U^{42} = I$
27	$U^{1023} = U$	$U^{512} = U$	$U^{1024} = U^2$	$U^{512} = U$

28	$U^{60} = U^4$	$U^{30} = U^2$	$U^{56} = I$	$U^{28} = I$
29	$U^{32767} = U$	$U^{16384} = U$	$U^{32766} = I$	$U^{16383} = I$
30	$U^{62} = U^2$	$U^{31} = U$	$U^{64} = U^4$	$U^{32} = U^2$
31	$U^{63} = U$	$U^{32} = U$	$U^{62} = I$	$U^{31} = I$
32	$U^{32} = 0$	$U^{16} = 0$	$U^{32} = I$	$U^{16} = I$

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